

SHARP L^2 BOUNDS FOR OSCILLATORY INTEGRAL OPERATORS WITH C^∞ PHASES

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1. INTRODUCTION

Consider the oscillatory integral operator on $L^2(\mathbb{R})$ of the form

$$(1.1) \quad Tf(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi(x,y) f(y) dy$$

with a C^∞ real phase $S(x,y)$ and a C^∞ cut-off $\chi(x,y)$ compactly supported in a small neighborhood of the origin in \mathbb{R}^2 . We are interested in the decay rate of the norm of T on $L^2(\mathbb{R})$ for $\lambda \rightarrow \infty$.

A well-known result of Hörmander [6] says that if the mixed partial derivative $S''_{xy} \neq 0$ on the support of χ , then the best possible estimate $\|T\| \leq C\lambda^{-1/2}$ is true. The problem of finding the optimal decay rate of $\|T\|$ for vanishing S''_{xy} has recently attracted much attention, especially because of its connection with the smoothing properties of generalized Radon transforms.

To describe the known results, we need some definitions. Let $\mathbb{Z}_+ \subset \mathbb{R}_+$ be the sets of non-negative integers and reals. The *Newton polygon of a set* $K \subset \mathbb{Z}_+^2$ is defined as the convex hull in \mathbb{R}_+^2 of the set $\bigcup_{n \in K} (n + \mathbb{R}_+^2)$.

Let

$$(1.2) \quad S''_{xy}(x,y) \sim \sum_{n \in \mathbb{Z}_+^2} c_n x^{n_1} y^{n_2}$$

be the formal Taylor expansion of S''_{xy} at the origin. The Newton polygon of the set $\{n \in \mathbb{Z}_+^2 | c_n \neq 0\}$ is called the *Newton polygon of the function* S''_{xy} and is denoted $\Gamma(S''_{xy})$.

Assume that $\Gamma(S''_{xy})$ is not empty, which means that the formal Taylor series of S''_{xy} is not the zero series. Denote by t_0 the parameter of intersection of the line $n_1 = n_2 = t$ with the boundary of $\Gamma(S''_{xy})$. The number $\delta = 1/(1 + t_0)$ is called the *Newton decay rate* of $S(x,y)$.

This quantity was introduced by Phong and Stein [7], who realized that the decay estimates for T depend on the Newton polygon of S''_{xy} and proved the bound

$$(1.3) \quad \|T\| \leq C\lambda^{-\delta/2}$$

under the additional assumption that $S(x,y)$ is real-analytic. They also showed that this bound is sharp in the sense that

$$\|T\| \geq C'\lambda^{-\delta/2}$$

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for large λ if $\chi(0,0) \neq 0$, and this part of their result does not depend on real analyticity.

On the other hand, the best known estimate in the general C^∞ case was

$$(1.4) \quad \|T\| \leq C_\varepsilon \lambda^{-\delta/2+\varepsilon},$$

which is implicitly contained in Seeger [9],[10]. It is interesting that the estimates (1.3) and (1.4) have been obtained by quite different arguments.

The purpose of our paper is to show that there is no loss of ε in the C^∞ case with one possible exception, when one loses at most a power of \log .

We say that S''_{xy} is *completely degenerate*, if its formal Taylor series (1.2) factorizes in the ring of formal power series $\mathbb{R}[[x,y]]$ as

$$U(x,y)(y-f(x))^N,$$

where $N \geq 2$, the series $f(x) \in \mathbb{R}[[x]]$ is of the form $f(x) = cx + \dots$ with $c \neq 0$, and the series $U(x,y) \in \mathbb{R}[[x,y]]$ is invertible. Note that $\delta = \frac{2}{N+2}$ for such a phase function. Then we have

Theorem 1.1. *There exists a small neighborhood of the origin V such that*

- (a) *If S''_{xy} is not completely degenerate, and $\text{supp } \chi \subset V$, then $\|T\| \leq C\lambda^{-\delta/2}$.*
- (b) *If S''_{xy} is completely degenerate, and $\text{supp } \chi \subset V$, then*

$$(1.5) \quad \|T\| \leq C\lambda^{-\frac{1}{N+2}} (\log \lambda)^{\frac{2N}{N+2}}$$

The rest of the paper is devoted to the proof of this theorem, which is based on a mixture of ideas from the above-mentioned works of Phong and Stein, and Seeger. Let us describe the scheme of the proof.

The geometry of the problem is best understood in terms of the singular variety $\mathcal{Z} = \{(x,y) | S''_{xy} = 0\}$. Similarly to what Phong and Stein have done in the real analytic case, we parameterize \mathcal{Z} by means of asymptotic Puiseux series (Section 2). To be more exact, assume for the purposes of this discussion that S''_{xy} has the form of a polynomial in y with C^∞ coefficients depending on x . (In fact, this case already contains all the essential difficulties.) Then we prove that for small x there exists a factorization

$$(1.6) \quad S''_{xy}(x,y) = \prod_{i=1}^n (y - y_i(x)),$$

where $y_i(x)$ are continuous \mathbb{C} -valued functions having asymptotic fractional power series expansions at zero. Now \mathcal{Z} splits into n branches $y = y_i(x)$ (see Fig. 1.1). In this discussion we assume for simplicity that all these branches are real.

To estimate the norm of T , we decompose the operator into pieces corresponding to the decomposition of the (x,y) plane into dyadic rectangles, so that $T = \sum T_{jk}$, where the kernel of T_{jk} is supported in $x \approx 2^{-j}$, $y \approx 2^{-k}$.

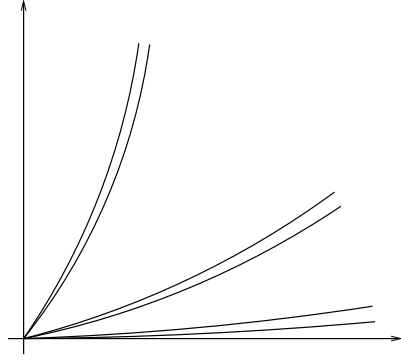


FIGURE 1.1

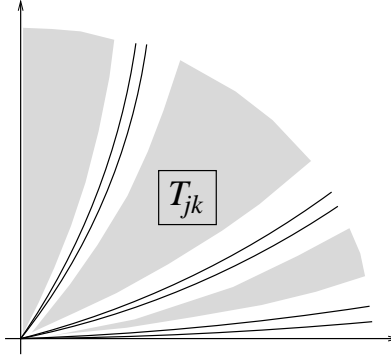


FIGURE 1.2

The way we treat a particular operator T_{jk} depends on its position with respect to \mathcal{Z} . The most simple and standard case is the one of all T_{jk} such that the distance from \mathcal{Z} to $\text{supp } T_{jk}$ is relatively large compared to the size of the support. The union of supports of such T_{jk} comprises the shaded regions in Fig. 1.2. For each of these T_{jk} we have from (1.6) a good lower bound on S''_{xy} on its support, which gives a good estimate on $\|T_{jk}\|$. Then the required norm estimate for the sum $\sum T_{jk}$ over all such T_{jk} is obtained with the help of a resummation procedure of Phong and Stein, which is based on almost orthogonality considerations (Section 3).

After that we are left with the T_{jk} supported near \mathcal{Z} . In Section 4 we estimate the contribution $T_x = \sum T_{jk}$ of the operators T_{jk} supported near the branches of \mathcal{Z} which are infinitely tangent to the x -axis (the corresponding functions $y_i(x)$ have the zero asymptotic expansion). Assume that there are exactly N such branches. We represent T_x as $T_x = \sum T_j$, where $T_j = \sum_k T_{jk}$ (see Fig. 1.3). Taking into account the fact that the derivative $\partial_y^N S''_{xy}$ does not vanish on the support of T_j , it is easy to obtain by van der Corput's lemma that the kernel of the operator $T_j T_j^*$ has the estimate

$$|K(x, y)| \leq C(\lambda|x - y|)^{-1/(N+1)}.$$

From this we can conclude that $\|T_j\| \leq C\lambda^{-\delta/2}$. Additional considerations show that the sum $\sum T_j$ is almost orthogonal in the sense that $\|T_j T_{j'}\| \leq C\lambda^{-\delta} 2^{-\varepsilon|j-j'|}$. Now the required estimate for $\|T_x\|$ follows by the Cotlar-Stein lemma.

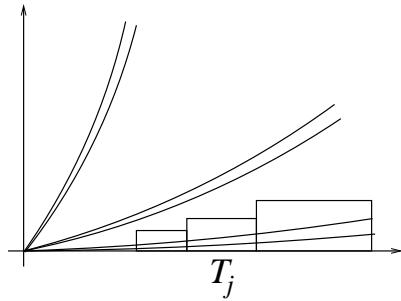


FIGURE 1.3

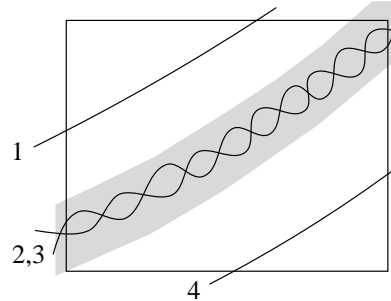


FIGURE 1.4

The most difficult part of the proof is the estimation of T_{jk} supported near the branches of \mathcal{Z} having nonzero asymptotic expansion. It is easy to see by almost orthogonality that each of these T_{jk} can be treated separately.

A typical geometric situation is shown in Fig. 1.4: we will have several branches of \mathcal{Z} intersecting the support of T_{jk} . Actually, we show that branches which are simple, i.e. have the asymptotic expansion different from asymptotic expansions of all the other branches, are given by functions $y_i(x)$ which are C^∞ away from the origin (branches 1, 4 in Fig. 1.4). On the other hand, multiple branches (2, 3 in Fig. 1.4) may not even be differentiable.

In Section 5, we isolate the multiple branches by a narrow cut-off (shaded area in Fig. 1.4). In the rest of $\text{supp } T_{jk}$, we take a Whitney-type decomposition into dyadic rectangles of the size $\delta \times L\delta$, where δ is comparable to the distance from the rectangle to \mathcal{Z} in the anisotropic metric $|x| + L^{-1}|y|$. Here L is determined by the condition $y'_i(x) \approx L$ for the C^∞ branches on $\text{supp } T_{jk}$.

On each Whitney rectangle we have good control over S''_{xy} , which leads to an estimate for the corresponding part of T_{jk} . On the other hand, we show that the rectangles with fixed δ form an almost orthogonal family. This fact is used to obtain the optimal norm estimate for the part of T_{jk} supported away from the multiple branches. We believe that this argument is simpler than the inductive procedure of separating the branches applied in a similar situation by Phong and Stein.

In Section 6, we deal with the part of T_{jk} supported near the multiple branches of \mathcal{Z} , i.e. in the shaded region in Fig. 1.4. Here we apply Seeger's method [9] with certain improvements possible in our case, and obtain the norm estimate $\lambda^{-1/(N+2)}(\log \lambda)^{2N/(N+2)}$, where N is the multiplicity of the branch ($N = 2$ in Fig. 1.4). This is exactly what is claimed in (1.5) if S''_{xy} is completely degenerate. If it is not, we show that this estimate is even better than $\lambda^{-\delta/2}$. This finishes the proof of the theorem.

Remark. By a completely different elementary proof based on a stopping-time argument, we can prove that for $N = 2$ the estimate (1.5) can be improved to the optimal $\|T\| \leq C\lambda^{-1/4}$. However, we do not know whether a similar improvement is possible for $N \geq 3$.

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2. FACTORIZATION OF C^∞ FUNCTIONS

Recall that if R is a ring, $R[t]$ and $R[[t]]$ are the rings of polynomials and, respectively, formal power series in t with coefficients from R . This notation can be iterated, e.g. $R[[x]][y]$ is the ring of polynomials in y with coefficients which are elements of $R[[x]]$, and $R[[x, y]]$ is the ring of double formal power series.

Our aim in this section will be to prove certain factorization formulas for C^∞ function, which will be valid in a small neighborhood of the origin. Since we do not care how small this neighborhood is, it will be convenient to formulate our results for *function-germs* rather than functions. An identity involving several function-germs is defined to be true if there exist functions from the equivalence classes of these germs such that in the intersection of their domains of definition the identity is true in the usual sense.

We will make use of the following rings of germs of \mathbb{C} -valued functions:

- $C((x))$ — continuous functions at the origin of \mathbb{R} ;

- $C^\infty((x))$ and $C^\infty((x, y))$ — C^∞ functions at the origin of \mathbb{R} and \mathbb{R}^2 , respectively;
- $C_+((x))$ and $C_+^\infty((x))$ — rings of one-sided germs; consist of (the equivalence classes of) functions $f(x)$ defined in a left half-neighborhood of zero of the form $[0, \varepsilon)$, where $\varepsilon > 0$ can depend on $f(x)$, which are continuous, respectively C^∞ , up to zero;

For the elements of $C^\infty((x, y))$, $C^\infty((x))$, and $C_+^\infty((x))$, we can talk about their Taylor series at the origin. A germ whose Taylor series is zero is called *flat*.

- $A_+((x^\gamma))$, $\gamma > 0$, — the subring of $C_+((x))$ consisting of germs $f(x)$, for which there exists a series $\bar{f}(x) \in \mathbb{C}[[x^\gamma]]$, $\bar{f}(x) = \sum_{n=0}^\infty c_n x^{n\gamma}$, such that $f(x) \sim \bar{f}(x)$ in the sense that for any N

$$f(x) - \sum_{n=0}^N c_n x^{n\gamma} = O(x^{(N+1)\gamma}), \quad x \rightarrow 0.$$

Such an $\bar{f}(x)$ is uniquely determined and is called the *asymptotic expansion* of $f(x)$.

The rings of germs of \mathbb{R} -valued functions will be denoted by adding an \mathbb{R} to the above notation, e.g. $\mathbb{R}C^\infty((x, y))$.

The main result of this section is the following

Proposition 2.1. *Let $F(x, y) \in \mathbb{R}C^\infty((x, y))$, $\Gamma = \Gamma(F)$ be the Newton polygon of $F(x, y)$, and assume that $\Gamma \neq \emptyset$. Let α run through all compact edges of the boundary of Γ . For each edge α joining integer points (A_α, B_α) and (A'_α, B'_α) , where $B'_\alpha > B_\alpha$, put $n_\alpha = B'_\alpha - B_\alpha$ and $\gamma_\alpha = (A_\alpha - A'_\alpha)/(B'_\alpha - B_\alpha)$. Let also A be the x -coordinate of the vertical infinite edge, and B be the y -coordinate of the horizontal infinite edge of Γ . Then the germ $F(x, y)$ admits in the region $x, y > 0$ a factorization of the form*

$$(2.1) \quad F(x, y) = U(x, y) \prod_{i=1}^A (x - X_i(y)) \prod_{i=1}^B (y - Y_i(x)) \prod_{\alpha} \prod_{i=1}^{n_\alpha} (y - Y_{\alpha i}(x)),$$

where

- (1) $U(x, y) \in \mathbb{R}C^\infty((x, y))$, $U(0, 0) \neq 0$,
- (2) all $X_i(x), Y_i(x) \in C_+((x))$, and $X_i(x), Y_i(x) = O(x^N)$ as $x \rightarrow 0$ for any $N > 0$,
- (3) all $Y_{\alpha i}(x) \in A_+((x^{1/n!}))$ for $n = B + \sum_{\alpha} n_{\alpha}$ with asymptotic expansions of the form $Y_{\alpha i}(x) = c_{\alpha i} x^{\gamma_{\alpha}} + \dots$ as $x \rightarrow 0$, where $c_{\alpha i} \neq 0$,
- (4) if $Y(x)$ is any of the functions $Y_{\alpha i}(x)$, and if $f(x, y) = \prod (y - Y_{\alpha i}(x))$ is the product over all i such that $Y_{\alpha i}(x)$ has exactly the same asymptotic expansion as $Y(x)$, then $f(x^{n!}, y) \in C_+^\infty((x))[y]$,
- (5) if in (4) we additionally assume that the asymptotic expansion of $Y(x)$ is real, then $f(x, y)$ is also real.

The proof relies on the following result, which is well known in the theory of plane algebraic curves as the *Puiseux theorem* (see [12], p. 98ff, or [1], A.V.150).

Lemma 2.2. *Let $\bar{F}(x, y) \in \mathbb{C}[[x]][y]$ be of the form*

$$\bar{F}(x, y) = y^n + \bar{c}_{n-1}(x)y^{n-1} + \dots + \bar{c}_0(x), \quad \bar{c}_i(x) \in \mathbb{C}[[x]],$$

where the zeroth order terms of all $c_i(x)$ vanish. Let α , n_α , γ_α , B be defined via the Newton polygon $\Gamma = \Gamma(\bar{F})$ in the same way as in Proposition 2.1. Then there

exists a factorization

$$(2.2) \quad \overline{F}(x, y) = y^B \prod_{\alpha} \prod_{i=1}^{n_{\alpha}} (y - \overline{Y}_{\alpha i}(x)),$$

where the series $\overline{Y}_{\alpha i}(x) \in \mathbb{C}[[x^{1/n!}]]$ are of the form $\overline{Y}_{\alpha i}(x) = c_{\alpha i} x^{\gamma_{\alpha}} + \dots$ with $c_{\alpha i} \neq 0$.

The following lemma will be used to get factorizations of function-germs from factorizations of their formal Taylor series.

Lemma 2.3. *Let $P(x, y) \in C^{\infty}((x))[y]$ be of the form*

$$P(x, y) = y^n + c_{n-1}(x)y^{n-1} + \dots + c_0(x), \quad c_i(x) \in C^{\infty}((x)),$$

where $c_i(0) = 0$ for all i . Let $\overline{P}(x, y) \in \mathbb{C}[[x]][y]$ be the formal Taylor series of $P(x, y)$ at the origin. Let $\overline{Y}(x) \in \mathbb{C}[[x]]$ be a root of multiplicity m , $1 \leq m \leq n$, of $\overline{P}(x, y)$ considered as a polynomial in y , i.e.

$$\overline{P}(x, \overline{Y}(x)) = \dots = \overline{P}_y^{(m-1)}(x, \overline{Y}(x)) = 0, \quad \overline{P}_y^{(m)}(x, \overline{Y}(x)) \neq 0$$

as elements of $\mathbb{C}[[x]]$. Then there exist m function-germs $Y_1(x), \dots, Y_m(x) \in C((x))$ such that

- (1) all $Y_i(x) \sim \overline{Y}(x)$ as $x \rightarrow 0$,
- (2) all $P(x, Y_i(x)) = 0$ for $x > 0$,
- (3) $\prod_{i=1}^m (y - Y_i(x)) \in C^{\infty}((x))[y]$,
- (4) if we additionally assume that $P(x, y)$ and $\overline{Y}(x)$ are real, then $\prod_{i=1}^m (y - Y_i(x))$ is also real.

Proof. Let $\tilde{Y}(x)$ be a C^{∞} function with the formal Taylor series $\overline{Y}(x)$, supplied by E. Borel's theorem. Denote

$$\delta_i(x) = \frac{1}{i!} P_y^{(i)}(x, \tilde{Y}(x)), \quad i = 0, \dots, n.$$

Let the (nonzero by assumption) series $\overline{P}_y^{(m)}(x, \overline{Y}(x))$ starts with a term cx^s , $c \neq 0$, $s \in \mathbb{Z}_+$. Then we have $\delta_m(x) = cx^s + o(x^s)$ as $x \rightarrow 0$. On the other hand, it is clear that $\delta_0(x), \dots, \delta_{m-1}(x)$ are flat.

We will be looking for $Y_i(x)$ of the form

$$Y(x) = \tilde{Y}(x) + \delta_m(x)\alpha(x),$$

where $\alpha(x)$ is an unknown continuous \mathbb{C} -valued function-germ such that $\alpha(x) = O(x^N)$ as $x \rightarrow 0$ for any $N > 0$.

By Taylor's formula, the equation $P(x, Y(x)) = 0$ can be written as

$$(2.3) \quad \sum_{i=0}^n \delta_i(x) [\delta_m(x)\alpha(x)]^i = 0.$$

For small x , this is equivalent to the equation $w(x, \alpha(x)) = 0$ for the function $w(x, z)$ given by

$$w(x, z) = \sum_{i=0}^n \delta_i(x) [\delta_m(x)]^{i-m-1} z^i.$$

Note that if $f, g \in C^{\infty}$, and f is flat at the origin, while g is not flat, then f/g is C^{∞} near the origin and is flat. It follows that $w(x, z) \in C^{\infty}((x))[z]$.

On the complex circle $|z| = x^N$, $N > 0$, the term z^m will dominate the other terms in $w(x, z)$ if x is sufficiently small. By Rouché's theorem it follows that the equation $w(x, z) = 0$ has for small fixed x exactly m roots in the disc $|z| < x^N$, which we denote $\alpha_i(x)$, $i = 1, \dots, m$. We can arrange so that $\alpha_i(x)$ are continuous in x , and the previous argument shows that $\alpha_i(x) = O(x^N)$ for any $N > 0$.

We now prove (3). Since the functions $\alpha_i(x)$ enter the product in (3) in a symmetric way, it is sufficient to prove that the elementary symmetric polynomials s_0, \dots, s_m in $\alpha_i(x)$ are in $C^\infty((x))$. By the Newton relations (see [1], A.IV.70), it is sufficient to prove the same for the functions

$$p_k(x) = \sum_{i=1}^m [\alpha_i(x)]^k, \quad k = 1, \dots, m.$$

However, by Cauchy's formula we have that for small x

$$p_k(x) = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{z^k w'_z(x, z)}{w(x, z)} dz,$$

from where it is clear that $p_k(x) \in C^\infty((x))$.

To prove (4), we notice that under the additional assumption made we can take $\tilde{Y}(x)$ to be real. Then $w(x, z) \in \mathbb{R}C^\infty((x))[z]$, and therefore non-real roots $\alpha_i(x)$ will appear in conjugate pairs. Then all $p_k(x)$ will be real, which implies (4). \square

Now we can prove

Lemma 2.4. *Proposition 2.1 is true if $F(x, y) \in \mathbb{R}C^\infty((x))[y]$.*

Proof. By Lemma 2.2, the Taylor series $\overline{F}(x, y) \in \mathbb{R}[[x]][y]$ of $F(x, y)$ has a factorization (2.2). Consider the function $P(x, y) = F(x^{n!}, y)$. Its Taylor series has the form $\overline{P}(x, y) = \overline{F}(x^{n!}, y)$, and so factorizes as

$$\overline{P}(x, y) = y^B \prod_{\alpha} \prod_{i=1}^{n_{\alpha}} (y - \overline{Y}_{\alpha i}(x^{n!})).$$

Let $\overline{Y}(x)$ be one of the series $\overline{Y}_{\alpha i}(x^{n!}) \in \mathbb{C}[[x]]$, and assume that among all the $\overline{Y}_{\alpha i}(x^{n!})$ there are exactly m series coinciding with $\overline{Y}(x)$. Then $y = \overline{Y}(x)$ is a root of multiplicity m of the polynomial $\overline{P}(x, y) \in \mathbb{R}[[x]][y]$, and by Lemma 2.3 we conclude that there exist m functions $Y_i(x) \in C((x))$, $i = 1, \dots, m$, such that (1)–(3) from the formulation of the lemma are true.

In view of (3), we can divide $P(x, y)$ by $\prod (y - Y_i(x))$, and the result is again a polynomial $\tilde{P}(x, y)$ from $C^\infty((x))[y]$. The Taylor polynomial of $\tilde{P}(x, y)$ will be $\overline{P}(x, y)$ divided by $(y - \overline{Y}(x))^m$. Now we can apply Lemma 2.3 to $\tilde{P}(x, y)$ choosing a different $\overline{Y}(x)$ etc.

By repeating this operation several times, we get a complete factorization of $P(x, y)$. The required factorization of $F(x, y)$ is then obtained by the inverse substitution $x \mapsto x^{1/n!}$. The property (5) is ensured by splitting off all real series $\overline{Y}(x)$ before non-real ones in the above argument. \square

Proposition 2.1 will be reduced to Lemma 2.4 by means of the following *Malgrange preparation theorem* (see [4], p. 95).

Lemma 2.5. *Let $F(x, y) \in \mathbb{R}C^\infty((x, y))$, and assume that $F(0, y)$ is not flat, so that $F(0, y) = cy^n + o(y^n)$, $y \rightarrow 0$, for some $n \in \mathbb{Z}_+$ and $c \neq 0$. Then there is a factorization*

$$F(x, y) = U(x, y)P(x, y),$$

where

- (1) $U(x, y) \in \mathbb{R}C^\infty((x, y))$, $U(0, 0) \neq 0$,
- (2) $P(x, y) \in \mathbb{R}C^\infty((x))[y]$ is of the form

$$P(x, y) = y^n + c_{n-1}(x)y^{n-1} + \dots + c_0(x),$$

where all $c_i(x) \in \mathbb{R}C^\infty((x))$, $c_i(0) = 0$.

Proof of Proposition 2.1. Notice that the Newton polygon is invariant with respect to multiplication by a nonzero C^∞ function (see Phong and Stein [7], p. 112). Therefore, for the functions $F(x, y)$ such that $F(0, y)$ is not flat (which is equivalent to having $A = 0$) the proposition follows immediately from Lemmas 2.5 and 2.4.

Assume now that $A > 0$. In this case we must somehow separate the roots infinitely tangent to the y -axis. This can be done as follows. Since $F(x, y)$ is not flat at the origin, there exists a rotated orthogonal system of coordinates (x', y') such that the restriction of F to the y' -axis is not flat. So we can apply Lemma 2.5 to F written in coordinates (x', y') . Let $P(x', y')$ be the arising polynomial.

If $y' = ax'$ is the equation of the old y -axis in the new coordinates, then $y' = ax'$ will be a root of multiplicity A of $\bar{P}(x', y') \in \mathbb{R}[[x']][y']$. So we can apply Lemma 2.3 and obtain A roots $y' = Y_i(x')$, $i = 1, \dots, A$, of $P(x', y') = 0$, such that $Y_i(x') \sim ax'$.

Moreover, by Lemma 2.3 (3),(4) we will have that $Q(x', y') = \prod(y' - Y_i(x'))$ is in $\mathbb{R}C^\infty((x'))[y']$. So we can divide $P(x', y')$ by $Q(x', y')$, and the quotient will be a C^∞ function, which is no longer flat on the old y -axis.

Let $\tilde{F}(x, y)$ be this last quotient written in the old system of coordinates. Then $\Gamma(\tilde{F})$ is just $\Gamma(F)$ shifted A units to the left. So we can factorize $\tilde{F}(x, y)$ as in the case $A = 0$ described above.

It remains to get a factorization of $Q(x', y')$ in the old coordinates. It is clear that the Taylor series of Q written in the coordinates (x, y) consists of one term cx^A . Interchanging the roles of x and y brings us back to the case $A = 0$, and the required factorization of the form $\prod(x - X_i(y))$ can be obtained as described above. \square

3. DYADIC DECOMPOSITION OF T . ESTIMATES AWAY FROM \mathcal{Z} .

We decompose the operator T as

$$T = \sum_{\pm} \sum_{j,k} T_{jk}^{\pm\pm},$$

where T_{jk}^{++} is defined as

$$T_{jk}^{++}f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi_j(x) \chi_k(y) \chi(x, y) f(y) dy.$$

Here $\sum_j \chi_j(x) = 1$ is a smooth dyadic partition of unity on \mathbb{R}^+ , so that the kernel of T_{jk}^{++} is supported on the rectangle $R_{jk} = [2^{-j-1}, 2^{-j+1}] \times [2^{-k-1}, 2^{-k+1}]$. Three other \pm combinations refer to the quadrants defined by specific signs of x and y . We

restrict ourselves with the positive quadrant, the other ones being exactly similar, and denote T_{jk}^{++} by simply T_{jk} .

Denote $F(x, y) = S_{xy}''(x, y)$. By Lemma 2.1, there is a neighborhood of the origin V such that in $V \cap \mathbb{R}_+^2$ there exists a factorization of the form (2.1). We assume that $\text{supp } \chi \subset V$. The singular variety $\mathcal{Z} = \{(x, y) \in V : F(x, y) = 0\}$ now splits into branches corresponding to the factors in the RHS of (2.1) (see Fig. 1.1). Note, however, that some of these branches may contain an imaginary component.

Let R_{jk}^* denote the double of R_{jk} . We fix a large constant D such that if the pair (j, k) satisfies the condition $\min_\alpha |k - j\gamma_\alpha| \geq D$, then $y - c_{\alpha i} x^{\gamma_\alpha} \neq 0$ on R_{jk}^* for all $c_{\alpha i} x^{\gamma_\alpha}$ occurring as the lowest order terms of the asymptotic expansions of $Y_{\alpha i}(x)$ in Proposition 2.1(3).

Let us number the compact edges α of the boundary of the Newton polygon $\Gamma(F)$ from left to right, so that $\gamma_1 < \gamma_2 < \dots < \gamma_{\alpha_0}$, where α_0 is the total number of compact edges. Also put $\gamma' = \gamma_1/2$ if $A > 0$, $\gamma' = 0$ otherwise; $\gamma'' = 2\gamma_{\alpha_0}$ if $B > 0$, $\gamma'' = \infty$ otherwise.

Consider the operators

$$(3.1) \quad T_\nu = \sum_{j\gamma_\nu \ll k \ll j\gamma_{\nu+1}} T_{jk}, \quad 1 \leq \nu \leq \alpha_0 - 1,$$

$$T' = \sum_{j\gamma' \ll k \ll j\gamma_1} T_{jk}, \quad T'' = \sum_{j\gamma_{\alpha_0} \ll k \ll j\gamma''} T_{jk},$$

where $a \ll b$ stands for $a \leq b - D$. These operators constitute the part of T supported relatively far away from \mathcal{Z} (see Fig. 1.2). In this section, we prove that $\|T_\nu\| \leq C\lambda^{-\delta/2}$ for each ν . The reader will believe us that with minor modifications the argument given below will also produce the same estimate for T', T'' .

Lemma 3.1. *Let T be an oscillatory integral operator of the form (1.1). Assume that*

- (1) $\chi(x, y)$ is supported in a rectangle R of size $\delta_x \times \delta_y$,
- (2) $|\partial_y^n \chi| \leq C/\delta_y^n$ in R for $n = 0, 1, 2$,
- (3) $|S_{xy}''| \geq \mu > 0$ in R ,
- (4) $|\partial_y^n S_{xy}''| \leq C\mu/\delta_y^n$ in R for $n = 1, 2$.

Then $\|T\| \leq \text{const}(\lambda\mu)^{-1/2}$ with const depending only on C .

This is a variant of the *Operator van der Corput lemma* of Phong and Stein [7]. The lemma is proved by a standard TT^* argument. The assumptions made are enough to show, integrating by parts twice, that the kernel of TT^* has the bound

$$K(x_1, x_2) \leq C_A \frac{\delta_y}{1 + \lambda^2 \mu^2 \delta_y^2 |x_1 - x_2|^2},$$

which implies the necessary norm estimate. We omit the details.

We will need the following partial case of the Schur test (see [5], Theorem 5.2).

Lemma 3.2. *Let T be an integral operator on $L^2(\mathbb{R})$ with kernel $K(x, y)$,*

$$Tf(x) = \int_{-\infty}^{\infty} K(x, y)f(y) dy.$$

Assume that the quantities

$$M_1 = \sup_y \int |K(x, y)| dx, \quad M_2 = \sup_x \int |K(x, y)| dy$$

are finite. Then T is bounded with $\|T\| \leq (M_1 M_2)^{1/2}$.

Corollary 3.3. (Phong and Stein [8], Lemma 1.6) *Let T be an integral operator with kernel $K(x, y)$, and assume that*

- (1) $|K(x, y)| \leq 1$,
 - (2) for each y , $K(x, y)$ is supported in an x -set of measure $\leq \delta_x$,
 - (3) for each x , $K(x, y)$ is supported in a y -set of measure $\leq \delta_y$.
- Then $\|T\| \leq (\delta_x \delta_y)^{1/2}$.

It is natural to call the latter bound on $\|T\|$ the *size estimate*, as opposed to the *oscillatory estimate* supplied by Lemma 3.1.

In the rest of the paper we will use the notation $a \lesssim b$ to mean $a \leq Cb$, and $a \approx b$ to mean $C^{-1}b \leq a \leq Cb$, where $C > 0$ is an unimportant constant, which is supposed to be independent of j, k, λ .

Take an operator T_{jk} entering the RHS of (3.1). We may reduce V if necessary so that on the part of \mathcal{Z} inside V the functions $Y_{\alpha i}$ do not differ much from the first terms of their asymptotic expansions. Assume that T_{jk} is nonzero, which means that $R_{jk} \cap V \neq \emptyset$. Then it is clear from the definition of D that the factors in the RHS of (2.1) can be estimated as follows for $(x, y) \in R_{jk}$:

$$\begin{aligned} |x - X_i(y)| &\approx 2^{-j}, \\ |y - Y_i(x)| &\approx 2^{-k}, \\ |y - Y_{\alpha i}(x)| &\approx \begin{cases} 2^{-k}, & \alpha > \nu, \\ 2^{-j\gamma_\alpha}, & \alpha \leq \nu. \end{cases} \end{aligned} \tag{3.2}$$

Therefore it follows from (2.1) that on R_{jk}

$$|F| \approx 2^{-jA} 2^{-kB} \prod_{\alpha > \nu} 2^{-kn_\alpha} \prod_{\alpha \leq \nu} 2^{-j\gamma_\alpha n_\alpha} =: \mu. \tag{3.3}$$

The numbers γ_α, n_α can be found from the Newton polygon $\Gamma(F)$ as described in Proposition 2.1. By using this information, we find that

$$\mu = 2^{-jA_\nu - kB_\nu}.$$

We claim that on R_{jk}

$$|\partial_y^n F| \lesssim \mu 2^{kn}, \quad n = 1, 2. \tag{3.4}$$

Indeed, when differentiating the RHS of (2.1) in y , the derivative can fall on either $U(x, y)$, or $\prod (x - X_i(y))$, or one of the remaining terms. In the first case, we simply get a bounded factor. In the second case, we get an even better factor of $O(2^{-kN})$ for any $N > 0$, since the product in question is a C^∞ function whose Taylor series at the origin is x^A . Finally, in the third case we get a factor of the form $(y - Y_i(x))^{-1}$ or $(y - Y_{\alpha i}(x))^{-1}$, which is $O(2^k)$ in view of (3.2). This argument works equally well for the second derivative, giving (3.4).

The rectangle R_{jk} is of size $\delta_x \times \delta_y$ with $\delta_x \approx 2^{-j}$, $\delta_y \approx 2^{-k}$. So the conditions of Lemma 3.1 are satisfied, and we obtain the oscillatory estimate

$$\|T_{jk}\| \lesssim \lambda^{-1/2} 2^{(jA_\nu + kB_\nu)/2}. \tag{3.5}$$

On the other hand, the size estimate following from Corollary 3.3 is

$$(3.6) \quad \|T_{jk}\| \lesssim 2^{(j+k)/2}.$$

The required bound for T_ν can now be derived from the last two estimates by summation, taking into account possible orthogonality relations between different T_{jk} . We use the summation procedure described in detail in Phong and Stein [7], pp. 120–122. Let us briefly recall what that procedure looks like.

We have three different cases: $A_\nu > B_\nu$, $A_\nu < B_\nu$, and $A_\nu = B_\nu$. If $A_\nu > B_\nu$, we put $k = [\gamma_\nu j] + r$, $r \geq -D$, substitute this into (3.5) and (3.6), and take the geometric mean of the two estimates killing the j -factor. The result is ([7], eq. (4.35))

$$\|T_{jk}\| \lesssim \lambda^{-\delta_\nu/2} 2^{-(1-\delta_\nu-B_\nu\delta_\nu)r/2},$$

where

$$(3.7) \quad \delta_\nu = \frac{1 + \gamma_\nu}{1 + A_\nu + (1 + B_\nu)\gamma_\nu}.$$

For a fixed r , the same estimate is true by almost orthogonality for the sum of T_{jk} over (j, k) satisfying $k = [\gamma_\nu j] + r$, since operators T_{jk} and $T_{j'k'}$ in such a sum have disjoint x - and y -supports for $|j - j'|$ larger than a fixed constant:

$$(3.8) \quad \left\| \sum_{k=[\gamma_\nu j]+r} T_{jk} \right\| \lesssim \lambda^{-\delta_\nu/2} 2^{-(1-\delta_\nu-B_\nu\delta_\nu)r/2},$$

In the case under consideration

$$1 - \delta_\nu - B_\nu\delta_\nu = \frac{A_\nu - B_\nu}{1 + A_\nu + (1 + B_\nu)\gamma_\nu} > 0,$$

so we can sum (3.8) in r , and obtain

$$\|T_\nu\| \leq \sum_{r=-D}^{\infty} \left\| \sum_{k=[\gamma_\nu j]+r} T_{jk} \right\| \lesssim \lambda^{-\delta_\nu/2}.$$

Let t_ν be the parameter of intersection of the line $n_1 = n_2 = t$ with the line containing edge ν of $\Gamma(F)$. Then a simple calculations shows that $1/\delta_\nu = 1 + t_\nu$. Therefore it is clear that $\delta_\nu \geq \delta$, and the previous estimate implies $\|T_\nu\| \lesssim \lambda^{-\delta/2}$, as desired.

Further, the case $A_\nu < B_\nu$ is analogous to the case $A_\nu > B_\nu$ and in fact reduces to it by interchanging the roles of j and k .

Finally, in the case $A_\nu = B_\nu$ we first sum the estimates (3.5) and (3.6) along the diagonals $j + k = i$. The corresponding sums are again almost orthogonal, and we get

$$\left\| \sum_{j+k=i} T_{jk} \right\| \lesssim \min(2^{-i/2}, \lambda^{-1/2} 2^{iA_\nu/2}),$$

whence

$$\|T_\nu\| \leq \sum_{i=0}^{\infty} \left\| \sum_{j+k=i} T_{jk} \right\| \lesssim \lambda^{-1/(2+2A_\nu)}.$$

Since in the considered case $1/(1+A_\nu) = \delta_\nu$, we again recover the required estimate.

The treatment of the operators T_ν is now complete.

4. ESTIMATES NEAR THE COORDINATE AXES.

In this section, we deal with the operators

$$T_y = \sum_{k \ll \gamma' j} T_{jk}, \quad T_x = \sum_{\gamma'' j \ll k} T_{jk}.$$

These two constitute the part of T supported near the branches of \mathcal{Z} which are infinitely tangent to the coordinate axes (see Fig. 1.3). We will prove the estimate $\|T_x\| \lesssim \lambda^{-\delta/2}$. The same estimate will be true for T_y , since taking the adjoint of T brings T_y to the form of T_x . We may of course assume $B \geq 1$, since otherwise $\gamma'' = \infty$ and $T_x = 0$.

We represent T_x as

$$T_x = \sum_j T_j, \quad T_j = \sum_{\gamma'' j \ll k} T_{jk},$$

and claim that

- (1) $\|T_j\| \lesssim \lambda^{-\delta/2}$,
- (2) $\|T_j^* T_{j'}\| = 0$ for $|j - j'| \geq 2$,
- (3) $\|T_j T_{j'}^*\| \lesssim \lambda^{-\delta/2} 2^{-\varepsilon|j-j'|}$ for some $\varepsilon > 0$.

If we prove all these, the estimate $\|T_x\| \lesssim \lambda^{-\delta/2}$ will follow from the Cotlar–Stein lemma.

We have

$$T_j f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi_j(x) \tilde{\chi}_j(y) \chi(x,y) f(y) dy,$$

where $\tilde{\chi}_j = \sum_{\gamma'' j \ll k} \chi_k$, so that the support of $\tilde{\chi}$ is contained in $[0, C2^{-\gamma'' j}]$. The property (2) is obvious. Further, the operator $T_j T_{j'}^*$ has the kernel

$$K(x_1, x_2) = \chi_j(x_1) \chi_{j'}(x_2) \int e^{i\lambda[S(x_1,y) - S(x_2,y)]} \tilde{\chi}_j(y) \tilde{\chi}_{j'}(y) \chi(x_1,y) \chi(x_2,y) dy.$$

We want to estimate this by the following variant of the standard van der Corput lemma (see [11], Corollary on p. 334).

Lemma 4.1. *Let k be a positive integer, $\Phi \in C^k[a, b]$, $\Psi \in C^1[a, b]$, and assume that $\Phi^{(k)} \geq \mu > 0$ on $[a, b]$. If $k = 1$, assume additionally that Φ' is monotonic on $[a, b]$. Then*

$$\left| \int_a^b e^{i\lambda\Phi(y)} \Psi(y) dy \right| \lesssim (\lambda\mu)^{-1/k} \left(|\Psi(b)| + \int_a^b |\Psi'| \right).$$

Assume that $j' \geq j$. We apply this lemma with $[a, b] = [0, C2^{-\gamma'' j}]$, $k = B + 1 \geq 2$,

$$\begin{aligned} \Phi(y) &= S(x_1, y) - S(x_2, y), \\ \Psi(y) &= \tilde{\chi}_j(y) \tilde{\chi}_{j'}(y) \chi(x_1, y) \chi(x_2, y). \end{aligned}$$

It is clear that $|\Psi(b)| + \int_a^b |\Psi'| \lesssim 1$. Further (recall that we denoted $S''_{xy} = F$),

$$\Psi^{(B+1)}(y) = \partial_y^{B+1} S(x_1, y) - \partial_y^{B+1} S(x_2, y) = \int_{x_2}^{x_1} \partial_y^B F(x, y) dx.$$

Of all the terms arising when we differentiate (2.1) B times in y , the term in which all derivatives fall on $\prod(y - Y_j(x))$ will dominate on the support of T_x after a possible reduction of V . It follows that on the support of T_x

$$|\partial_y^B F| \approx x^{A+\sum_\alpha n_\alpha \gamma_\alpha} = x^{A'},$$

where we denoted $A' = A_{\alpha_0}$. Note that (A', B) is the common vertex of the horizontal infinite edge of $\Gamma(F)$ and its last compact edge α_0 .

By the previous remarks,

$$|\Psi^{(B+1)}(y)| \gtrsim |x_1^{A'+1} - x_2^{A'+1}|.$$

In the case $j' = j$ we have $|x_1^{A'+1} - x_2^{A'+1}| \approx 2^{-jA'}|x_1 - x_2|$ on the support of $K(x_1, x_2)$, so Lemma 4.1 gives

$$|K(x_1, x_2)| \leq 2^{jA'/(B+1)}(\lambda|x_1 - x_2|)^{-1/(B+1)}.$$

So by Lemma 3.2,

$$\begin{aligned} \|T_j T_j^*\| &\lesssim 2^{jA'/(B+1)} \int_0^{2^{-j}} (\lambda t)^{-1/(B+1)} dt \\ &\lesssim \lambda^{-1/(B+1)} 2^{j(A'-B)/(B+1)}. \end{aligned}$$

This of course implies the estimate

$$(4.1) \quad \|T_j\| \lesssim \lambda^{-1/(2B+2)} 2^{j(A'-B)/(2B+2)}.$$

Another estimate is supplied by Corollary 3.3:

$$(4.2) \quad \|T_j\| \lesssim 2^{-j(1+\gamma'')/2} \leq 2^{-j(1+\gamma_{\alpha_0})/2}.$$

As the reader may check, taking the geometrical mean of these two bounds which kills the j -factor gives exactly $\|T_j\| \lesssim \lambda^{-\delta_{\alpha_0}/2}$, with δ_{α_0} defined as in (3.7). This implies (1) since all $\delta_\nu \geq \delta$.

In proving (3), we may assume $j' \geq j+2$. Then $|x_1^{A'+1} - x_2^{A'+1}| \approx 2^{-j(A'+1)}$ on the support of $K(x_1, x_2)$, whence by Lemma 4.1

$$|K(x_1, x_2)| \lesssim \lambda^{-1/(B+1)} 2^{j(A'+1)/(B+1)} =: M.$$

The support of $K(x_1, x_2)$ is contained in the rectangle of size $\approx 2^{-j} \times 2^{-j'}$. Now Corollary 3.3 gives the bound

$$\|T_j T_{j'}^*\| \lesssim M 2^{-(j+j')/2} = \lambda^{-1/(B+1)} 2^{j(A'-B)/(B+1)} 2^{-\Delta j/2},$$

where we denoted $\Delta j = j' - j$. By multiplying the estimates (4.2) for T_j and $T_{j'}$, we get another bound:

$$\|T_j T_{j'}^*\| \lesssim 2^{-j(1+\gamma_{\alpha_0})} 2^{-\Delta j(1+\gamma_{\alpha_0})/2}.$$

These two bounds have the form of (4.1) and (4.2) squared, but with an additional factor exponentially decreasing in Δj . Therefore it is clear that this time taking the geometric mean killing the j -factor will give

$$\|T_j T_{j'}^*\| \lesssim \lambda^{-\delta_{\alpha_0}} 2^{-\varepsilon \Delta j}$$

for some $\varepsilon > 0$. This implies (3) and concludes the treatment of T_x .

5. ESTIMATES NEAR \mathcal{Z} .

We still have to estimate the part of T supported near the branches of Z which are not infinitely tangent to the coordinate axes. This part is the sum over $\nu = 1, \dots, \alpha_0$ of the operators

$$(5.1) \quad T^\nu = \sum_{\gamma_\nu j - D < k < \gamma_\nu j + D} T_{jk}.$$

Notice that the sum in (5.1) is almost orthogonal, since the x - and y -supports of T_{jk} and $T_{j'k'}$ are disjoint for $|j - j'|$ larger than a fixed constant. Therefore it suffices to prove the estimate $\|T_{jk}\| \lesssim \lambda^{-\delta/2}$ for each T_{jk} from the RHS of (5.1).

Fix such a T_{jk} . Analogously to (3.3), on R_{jk}

$$(5.2) \quad \begin{aligned} |F| &\approx 2^{-jA} 2^{-j\gamma_\nu B} \prod_{\alpha > \nu} 2^{-j\gamma_\nu n_\alpha} \prod_{\alpha < \nu} 2^{-j\gamma_\alpha n_\alpha} \prod_{i=1}^{n_\nu} |y - Y_{\nu i}(x)| \\ &= 2^{-j(\gamma_\nu B_\nu + A_\nu - \gamma_\nu n_\nu)} \prod_{i=1}^{n_\nu} |y - Y_{\nu i}(x)| \\ &= 2^{-j(\gamma_\nu B_\nu + A_\nu - \gamma_\nu n'_\nu)} \prod_{i=1}^{n'_\nu} |y - Y_{\nu i}(x)|, \end{aligned}$$

where we ordered $Y_{\nu i}$ so that for $n'_\nu < i \leq n_\nu$ we have $\operatorname{Re} c_{\nu i} = 0$ in $Y_{\nu i} = c_{\nu i} x^{\gamma_\nu} + \dots$.

Let us quickly dispose of the case $n'_\nu = 0$, in which we can apply Lemma 3.1 (the condition (4) is easily checked) and Corollary 3.3 to get the oscillatory and size estimates

$$\begin{aligned} \|T_{jk}\| &\lesssim \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2}, \\ \|T_{jk}\| &\lesssim 2^{-j(1+\gamma_\nu)/2}. \end{aligned}$$

Now by taking the geometric mean killing the j -factor, we obtain the required estimate $\|T_{jk}\| \lesssim \lambda^{-\delta_\nu/2} \leq \lambda^{-\delta/2}$.

Now assume that $n'_\nu > 0$. Denote $r_i(x) = \operatorname{Re} Y_{\nu i}(x)$, and let $\bar{r}_i(x) \in \mathbb{R}[[x^{1/n!}]]$ be the asymptotic expansion of $r_i(x)$ at zero. By E. Borel's theorem, we can find real functions $f_i(x)$ such that $f_i(x^{n!}) \in C^\infty$ and $f_i(x) \sim \bar{r}_i(x)$ as $x \rightarrow 0$. Moreover, there is one case when we may and will take simply $f_i(x) = Y_{\nu i}(x)$. Namely, by Proposition 2.1 (4),(5) this is possible if the series $\bar{Y}_{\nu i}(x)$ is real and different from any other $\bar{Y}_{\nu i'}(x)$.

Let W be the union of the graphs of $f_i(x)$ inside R_{jk} :

$$W = \bigcup_{i=1}^{n'_\nu} \{(x, y) \in R_{jk} \mid y = f_i(x)\}.$$

It is not difficult to see that on R_{jk}

$$f'_i(x) \approx x^{\gamma_\nu - 1} \approx 2^{-j(\gamma_\nu - 1)} =: L.$$

This suggests to consider a Whitney-type decomposition of $R_{jk} \setminus W$ away from W into rectangles of the size $2^{-m} \times L 2^{-m}$. The easiest way to do this is to dilate the set $R_{jk} \setminus W$ along the y -axis L^{-1} times, take the standard Whitney decomposition

into the dyadic squares away from (the dilation of) W , and contract everything to the original scale. As a result, we get a covering

$$R_{jk} \setminus W \subset \bigcup R_l, \quad R_l \cap R_{jk} \neq \emptyset,$$

where R_l are rectangles of the size $2^{-m_l} \times L2^{-m_l}$, $m_l \in \mathbb{Z}_+$, such that the distance from R_l to W in the anisotropic norm $|x| + L^{-1}|y|$ is of the order 2^{-m_l} .

We claim that the rectangles R_l of *fixed* size form an almost orthogonal family, i.e. that for each R_l the number of rectangles $R_{l'}$ with $m_{l'} = m_l$ such that either the x - or the y -projections of R_l and $R_{l'}$ intersect is bounded by a fixed constant independent of l .

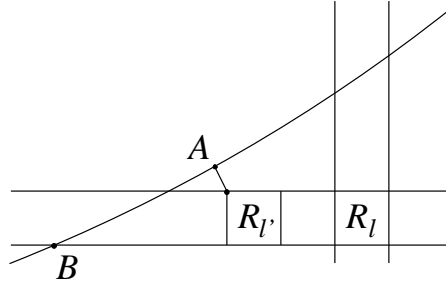


FIGURE 5.1

Consider the case of intersecting y -projections (the other case is similar). Then $R_{l'}$ is contained in the horizontal strip passing through R_l (see Fig. 5.1). By dilating along the y -axis, we may assume that $L = 1$. Since $\text{dist}(R_{l'}, W) \approx 2^{-m_l}$, there exists a point A on the graph of one of the functions $f_i(x)$ such that $\text{dist}(R_{l'}, A) \approx 2^{-m_l}$. Let B denote the point where the graph of $f_i(x)$ intersects the bottom of the strip. Since $f'_i(x) \approx L = 1$, we have $\text{dist}(A, B) \lesssim 2^{-m_l}$, and therefore $\text{dist}(R_{l'}, B) \lesssim 2^{-m_l}$. Thus all possible rectangles $R_{l'}$ are situated at a distance $\lesssim 2^{-m_l}$ from no more than n'_ν points where the bottom of the horizontal strip intersects W . This implies that the number of $R_{l'}$ in the horizontal strip is bounded by a fixed constant, and the almost orthogonality is verified.

Now let $R_l^* = (1 + \varepsilon)R_l$, where an $\varepsilon > 0$ is chosen so small that $\text{dist}(R_l^*, W) \approx 2^{-m_l}$ (in the anisotropic norm). Consider a smooth partition of unity $\sum_l \varphi_l = 1$ on $\bigcup R_l$ with $\text{supp } \varphi_l \subset R_l^*$, satisfying the natural differential inequalities. We are going to decompose T_{jk} using this partition of unity. However, this decomposition will not be useful near the real multiple branches of \mathcal{Z} , since we will not have good control on the size of F there. For now we are just going to localize away from those branches in the following way.

Let β_i denote the power exponent of the first nonzero term in the asymptotic expansion of $\text{Im } Y_{\nu i}(x)$; $\beta_i := \infty$ if this expansion is zero. For a large fixed number Q we introduce the set

$$W_Q = \bigcup_{i=1}^{n'_\nu} \{ (x, y) \in R_{jk} \mid |y - f_i(x)| \leq 2^{-jQ} \},$$

where $*$ indicates that the union is taken over all i such that $\beta_i = \infty$ and $f_i(x) \neq Y_{\nu i}(x)$. By the choice of $f_i(x)$, this may happen only if the series $\bar{Y}_{\nu i}(x)$ is real and there are several $\bar{Y}_{\nu i'}(x)$ having $\bar{Y}_{\nu i}(x)$ as their asymptotic expansion. One

can say that W_Q is the tubular neighborhood of width 2^{-jQ} of the real multiple branches of \mathcal{Z} (see Fig. 1.4).

The purpose of introducing W_Q is that on $R_{jk} \setminus W_Q$ we have (if j is large enough, which can be achieved by a further contraction of V)

$$(5.3) \quad |y - Y_{\nu i}(x)| \approx |y - f_i(x)| + 2^{-j\beta_i}.$$

Now let χ_Q be a smooth cut-off supported in the double of W_Q , $\chi_Q \equiv 1$ on W_Q . We consider the decomposition

$$(5.4) \quad \begin{aligned} T_{jk} &= T_Q + T^Q, \\ T_Q f(x) &= \int e^{i\lambda S(x,y)} \chi_Q(x,y) \chi_j(x) \chi_k(y) \chi(x,y) f(y) dy, \\ T^Q &= \sum_l T_l^Q, \\ T_l^Q f(x) &= \int e^{i\lambda S(x,y)} \varphi_l(x,y) (1 - \chi_Q(x,y)) \chi_j(x) \chi_k(y) \chi(x,y) f(y) dy, \end{aligned}$$

In the rest of this section we prove that $\|T^Q\| \lesssim \lambda^{-\delta/2}$. The operator T_Q will be dealt with in the next section.

Let T_l^Q be one of the operators from the decomposition of T^Q , and assume that $T_l^Q \neq 0$, i.e. that $R_l^* \cap (R_{jk} \setminus W_Q) \neq \emptyset$. Fix a point (x_l, y_l) in this last intersection. We claim that

$$(5.5) \quad |y - f_i(x)| \approx |y_l - f_i(x_l)|$$

for $(x, y) \in R_l^*$ and $i = 1, \dots, n'_\nu$. Indeed, let (x', y') and (x'', y'') be points of R_l^* for which the value of $|y - f_i(x)|$ is respectively minimal and maximal. Then

$$\begin{aligned} |y'' - f_i(x'')| &\leq |y' - f_i(x')| + |y'' - y'| + |f_i(x'') - f_i(x')| \\ &\lesssim |y' - f_i(x')| + L2^{-m_l} \\ &\lesssim |y' - f_i(x')|, \end{aligned}$$

since $L^{-1}|y' - f_i(x')| \geq \text{dist}(R_l^*, W) \gtrsim 2^{-m_l}$. From this (5.5) follows.

Now from (5.2) and (5.3) we see that on R_l^*

$$|F| \approx 2^{-j(\gamma_\nu B_\nu + A_\nu - \gamma_\nu n'_\nu)} \prod_{i=1}^{n'_\nu} (|y_l - f_i(x_l)| + 2^{-j\beta_i}) =: \mu_l.$$

It follows by Lemma 3.1 (the condition (4) needs to be checked, but this is easy) that $\|T_l^Q\| \lesssim (\mu_l \lambda)^{-1/2}$.

We can get a lower bound on μ_l by noting that $|y_l - f_i(x_l)| \gtrsim L2^{-m_l} = 2^{-j(\gamma_\nu - 1) - m_l}$. This gives

$$\mu_l \gtrsim 2^{-j(\gamma_\nu B_\nu + A_\nu)} 2^{(m_l - j)n'_\nu},$$

and therefore

$$(5.6) \quad \|T_l^Q\| \lesssim \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2} 2^{-(m_l - j)n'_\nu/2}.$$

On the other hand, by Corollary 3.3,

$$(5.7) \quad \|T_l^Q\| \lesssim 2^{-m_l - j(\gamma_\nu - 1)/2}.$$

It is clear that the passage from R_l to R_l^* preserved the almost orthogonality of rectangles with fixed m_l . Therefore (5.6) and (5.7) imply

$$\left\| \sum_{m_{l'}=m_l} T_{l'}^Q \right\| \lesssim \min \left(2^{-m_l-j(\gamma_\nu-1)/2}, \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2} 2^{-(m_l-j)n'_\nu/2} \right).$$

Simple size considerations show that only rectangles with $m_l \geq j - C$ may occur in the decomposition of T^Q . Therefore,

$$\begin{aligned} \|T^Q\| &\leq \sum_{m_l=j-C}^{\infty} \left\| \sum_{m_{l'}=m_l} T_{l'}^Q \right\| \\ &\lesssim \min \left(2^{-j(\gamma_\nu+1)/2}, \lambda^{-1/2} 2^{j(\gamma_\nu B_\nu + A_\nu)/2} \right). \end{aligned}$$

This is a familiar expression, and by taking the geometrical mean killing the j -factor we obtain $\|T^Q\| \lesssim \lambda^{-\delta_\nu/2} \leq \lambda^{-\delta/2}$.

6. ESTIMATES NEAR THE MULTIPLE REAL BRANCHES OF \mathcal{Z} .

To finish the proof of the theorem, we must estimate the operator T_Q appearing in the decomposition (5.4) of T_{jk} .

In the estimates below we can assume that $\gamma_\nu \geq 1$, since this can be achieved by passing to the adjoint operator if necessary.

Further, we can assume that Q is chosen so large that the branches of \mathcal{Z} having different asymptotic expansions become completely separated in the definition of W_Q . Since such branches can be treated separately, we are reduced to the case when T_Q has the form

$$\begin{aligned} T_Q f(x) &= \int e^{i\lambda S(x,y)} \chi_{jkQ}(x,y) f(y) dy, \\ \chi_{jkQ}(x,y) &= \chi_j(x) \chi_k(y) \varphi(2^{jQ}(y - g(x))). \end{aligned}$$

Here $\varphi(t)$ is a C^∞ cut-off supported in $[-1, 1]$, $g(x^{n!}) \in \mathbb{R}C^\infty$, $g(x) = cx^{\gamma_\nu} + \dots$, $c \neq 0$, and in the factorization (2.1) exactly $N \geq 2$ functions $Y_{\nu i}(x)$ have asymptotic expansion coinciding with that of $g(x)$. We will assume that this happens for $i = 1, \dots, N$. We also re-denote $W_Q = \{(x, y) \in R_{jk} \mid |y - g(x)| \leq 2^{-jQ}\}$.

We write $F(x, y)$ as

$$F(x, y) = \tilde{U}(x, y) P(x, y),$$

where $P(x, y) = \prod_{i=1}^N (y - Y_{\nu i}(x))$, and $\tilde{U}(x, y)$ is the product of the rest of the terms in (2.1).

Since all the branches of \mathcal{Z} appearing in $\tilde{U}(x, y)$ are well separated from W_Q , there exists a constant $M_1 \geq 0$ such that

$$|\tilde{U}| \approx 2^{-jM_1} \quad \text{on } W_Q.$$

Moreover, it can be seen directly that if S''_{xy} is completely degenerate, we have $M_1 = 0$.

Further, by Proposition 2.1 (4), (5) we know that $P(x, y) \in \mathbb{R}C_+^\infty((x^{1/n!}))[y]$, so that $P(x, y)$ is C^∞ in both variables on W_Q . It is clear that

$$(6.1) \quad \partial_y^N P(x, y) = \text{const} \neq 0.$$

We claim that, more generally,

$$(6.2) \quad \partial_x^k \partial_y^{N-k} P(x, y) \neq 0 \quad \text{on} \quad W_Q, \quad k = 0, \dots, N.$$

Denote $Q(x, y) = P(x^{n!}, y) \in C_+^\infty((x))[y]$. The Taylor series of $Q(x, y)$ is

$$\overline{Q}(x, y) = \prod_{i=1}^N (y - \overline{G}(x)), \quad \overline{G}(x) = \overline{g}(x^{n!}).$$

It is clear that

$$\begin{aligned} [\partial_x^l \partial_y^{N-k} \overline{Q}](x, \overline{G}(x)) &= 0, \quad 0 \leq l < k, \\ [\partial_x^k \partial_y^{N-k} \overline{Q}](x, \overline{G}(x)) &\neq 0. \end{aligned}$$

Therefore the factorizations of $\partial_x^l \partial_y^{N-k} Q(x, y)$, $l < k$, which can be obtained as described in the proof of Lemma 2.4, will contain branches with the asymptotic expansion $\overline{G}(x)$, while the factorization of $\partial_x^k \partial_y^{N-k} Q(x, y)$ will not contain such branches. This implies (6.2), provided that Q is large enough, since $\partial_x^k \partial_y^{N-k} P(x, y)$ can be expressed as

$$(\partial_x^k \partial_y^{N-k} Q)(x^{1/n!}, y) + \sum_{l < k} c_l(x) (\partial_x^l \partial_y^{N-k} Q)(x^{1/n!}, y)$$

with coefficients $c_l(x)$ growing power-like as $x \rightarrow 0$.

In addition, the above argument gives an estimate

$$(6.3) \quad \partial_x^N P(x, y) \geq 2^{-jM_2} \quad \text{on} \quad W_Q,$$

for some constant $M_2 \geq 0$; $M_2 = 0$ if S''_{xy} is completely degenerate.

Denote $\sigma_j(x, y) = \frac{1}{j!} \partial_y^j P(x, y)$. Consider the decomposition

$$\begin{aligned} T_Q &= \sum_{l=-C}^{\infty} T_l, \\ T_l f(x) &= \int e^{i\lambda S(x, y)} \chi_{jkQ}(x, y) \overline{\chi}_l(\sigma_0(x, y)) f(y) dy, \end{aligned}$$

where $\overline{\chi}_l(t)$ is the characteristic function of the set $2^{-l} \leq |t| \leq 2^{-l+1}$, C is a constant.

We are going to prove the estimates:

$$(6.4) \quad \|T_l\| \lesssim 2^{-l/N + jM_2/2N},$$

$$(6.5) \quad \|T_l\| \lesssim \lambda^{-1/2} (\log \lambda)^{1/2} 2^{l/2} l^{N-1/2} 2^{jM_1/2}.$$

The required bound for T_Q can then be derived as follows.

Consider first the completely degenerate case, when $M_1 = M_2 = 0$. We have

$$\|T_Q\| \lesssim \sum_{l=0}^{\infty} \min(2^{-l/N}, \lambda^{-1/2} 2^{l/2} (\log \lambda)^{1/2} l^{N-1/2}).$$

If it were not for the factor of $(\log \lambda)^{1/2} l^{N-1/2}$, the terms in parentheses would become equal for $l = l_0 = \frac{N}{N+2} \log \lambda$, and we would have the estimate $\|T_Q\| \lesssim$

$\lambda^{-1/(N+2)}$. In the present situation we put $l_0 = \frac{N}{N+2} \log \lambda - k \log \log \lambda$ with indeterminate k and have the estimate

$$\begin{aligned} \|T_Q\| &\lesssim 2^{-l_0/M} + \lambda^{-1/2} 2^{l_0/2} (\log \lambda)^{1/2} l_0^{N-1/2} \\ &\lesssim \lambda^{-1/(N+2)} \left[(\log \lambda)^{k/N} + (\log \lambda)^{N-k/2} \right]. \end{aligned}$$

The optimal value of k is $k = \frac{2N^2}{N+2}$, which gives

$$\|T_Q\| \lesssim \lambda^{-\frac{1}{N+2}} (\log \lambda)^{\frac{2N}{N+2}}$$

in complete accordance with what is claimed in the theorem.

Assume now that S''_{xy} is not completely degenerate. In this case the above argument gives in any case the estimate

$$\|T_Q\| \leq C_\varepsilon 2^{jM} \lambda^{-\frac{1}{N+2} + \varepsilon}$$

for any $\varepsilon > 0$, with some constant M . (We do not pursue the possibility of obtaining a log factor here, since as we will see in a moment, what we have is already good enough.) Further, by Corollary 3.3 we certainly have the estimate

$$\|T_Q\| \lesssim 2^{-jQ/2}.$$

The idea is that now we can take the geometric mean of the last two estimates killing the j -factor and, if Q is very large, this will introduce only a very small increase in the exponent of λ , actually tending to zero as $Q \rightarrow \infty$. Thus we have

$$\|T_{Q_\varepsilon}\| \leq C_\varepsilon \lambda^{-\frac{1}{N+2} + \varepsilon}.$$

We will show in a moment that in the case under consideration $1/(N+2) > \delta/2$. This allows us to choose and fix Q from the very beginning so large that $\|T_Q\| \lesssim \lambda^{-\delta/2}$, thus proving the theorem.

We show that in fact $1/(N+2) > \delta_\nu/2$. Indeed, since we already have N branches whose expansion starts with cx^{γ_ν} , we know that $n_\nu \geq N$. Therefore $A_\nu = n_\nu \gamma_\nu + A'_\nu \geq N \gamma_\nu$, and

$$\delta_\nu \leq \frac{1 + \gamma_\nu}{1 + N \gamma_\nu + \gamma_\nu} \leq \frac{2}{N+2},$$

since $\gamma_\nu \geq 1$. Besides that, the equality holds if and only if $\gamma_\nu = 1$, $A_\nu = N$, $B_\nu = 0$. But this corresponds exactly to the completely degenerate case, which is excluded.

We now turn to the proof of the claimed bounds for T_l . The proof of (6.4) is easy and is based on the following well-known

Lemma 6.1. (Christ [2], Lemma 3.3) *Let $f \in C^N[a, b]$ be such that $f^{(N)} \geq \mu > 0$ on $[a, b]$. Then for any $\gamma > 0$*

$$|\{x \in [a, b] : |f(x)| \leq \gamma\}| \leq A_k (\gamma \mu)^{1/k},$$

where the constant A_k depends only on k .

By this lemma, in view of (6.1) and (6.3), the kernel of T_l is supported in a y -set of measure $\lesssim 2^{-l/N}$ for each x , and in an x -set of measure $\lesssim 2^{-l/N + jM_2/N}$ for each y . Now (6.4) follows by Corollary 3.3.

The proof of (6.5) is more intricate and is carried out by a variation of a method developed in Seeger [9], Section 3. The key idea is to take an additional dyadic localization in σ_j , $1 \leq j \leq N-1$. Let l be fixed; all constants below will however

be independent of l . Let $\gamma = (\gamma_1, \dots, \gamma_{N-1})$ be a vector with integer components $-C \leq \gamma_i \leq l$, C some constant. Denote

$$\chi_\gamma(x, y) = \chi_{jkQ}(x, y) \bar{\chi}_l(\sigma_0(x, y)) \prod_{i=1}^{N-1} \bar{\chi}_{\gamma_i}(\sigma_i(x, y)),$$

where $\bar{\chi}_{\gamma_i}(t)$ is the characteristic function of the set $2^{-\gamma_i} \leq |t| \leq 2^{-\gamma_i+1}$ for $\gamma_i < l$, and of the set $|t| \leq 2^{-l+1}$ for $\gamma_i = l$.

For an appropriate fixed C we have the decomposition

$$T_l = \sum_{\gamma} T_{\gamma},$$

$$T_{\gamma} f(x) = \int e^{i\lambda S(x, y)} \chi_{\gamma}(x, y) f(y) dy.$$

We are going to prove that for each γ

$$(6.6) \quad \|T_{\gamma}\| \lesssim \lambda^{-1/2} (\log \lambda)^{1/2} 2^{l/2} l^{1/2} 2^{jM_1/2}.$$

This will imply (6.5), since the number of T_{γ} in the decomposition of T_l is $\lesssim l^{N-1}$.

The kernel of the operator $T_{\gamma}^* T_{\gamma}$ has the form

$$K(y_2, y_1) = \int e^{i\lambda[S(x, y_2) - S(x, y_1)]} \chi_{\gamma}(x, y_1) \chi_{\gamma}(x, y_2) dx.$$

Assuming that $y_2 > y_1$, and using Taylor's formula in y for $P(x, y)$, we have

$$(6.7) \quad [S(x, y_2) - S(x, y_1)]'_x = \int_{y_1}^{y_2} \tilde{U}(x, y) P(x, y) dy$$

$$= \int_{y_1}^{y_2} \tilde{U}(x, y) \left[\sum_{j=0}^N \sigma_j(x, y_1) (y - y_1)^j \right] dy$$

$$= \sum_{j=0}^N \sigma_j(x, y_1) \int_{y_1}^{y_2} \tilde{U}(x, y) (y - y_1)^j dy.$$

Notice that $\int_{y_1}^{y_2} \tilde{U}(x, y) (y - y_1)^j dy \approx 2^{-jM_1} (y_2 - y_1)^{j+1}$. So the RHS of (6.7) looks like a polynomial in $y_2 - y_1$ with dyadically restricted coefficients. To handle such polynomials, we need the following variant of Lemma 3.2 from [9]. We chose to give a proof, since we have found a one simpler than in [9].

Lemma 6.2. *For an integer $N \geq 1$, an integer vector $r = (r_1, \dots, r_N)$, $r_i \geq 0$, and a constant $C > 0$ consider the set $\mathcal{P} = \mathcal{P}(r, C, N)$ of all polynomials of the form $P(h) = 1 + \sum_{i=1}^N a_i h^i$ with real coefficients a_i satisfying*

$$|a_i| \in [C^{-1}2^{r_i}, C2^{r_i}] \quad \text{if } r_i > 0,$$

$$|a_i| \leq C \quad \text{if } r_i = 0.$$

Then there exists a constant $B = B(C, N)$, independent of r , and a set $E \in [0, 1]$ of the form

$$(6.8) \quad E = [0, 2^{\beta_1}] \cup [2^{\alpha_2}, 2^{\beta_2}] \cup \dots \cup [2^{\alpha_s}, 2^{\beta_s}],$$

such that

(1) α_i, β_i are integers, $\beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_s < \beta_s \leq 0$,

- (2) $s \leq B$; $\beta_1 \geq -B \max(r_i)$; $[(1 - \beta_s) + \sum_{j=1}^{s-1} (\alpha_{j+1} - \beta_j)] \leq B$,
 (3) $|P(h)| \geq B^{-1}$ for $h \in E$ for any $P \in \mathcal{P}$.

Proof. Put $r_0 = 0$. Consider the convex set Σ given as the intersection of the half-planes lying above the lines $y = r_i + ix$, $i = 0, \dots, N$. Let A_i , $i = 1, \dots, n$ be all the corner points of the boundary of Σ with the x -coordinates $x_1 < x_2 < \dots < x_n$. It is clear that $n \leq N$. We claim that for any $P \in \mathcal{P}$

$$|P(h)| \geq B^{-1} \quad \text{if } h > 0, \quad \log h \notin \bigcup_{j=1}^n (x_j - B, x_j + B).$$

It is not difficult to see that this implies (1)–(3).

Let $\log h \in [x_j + B, x_{j+1} - B]$, and assume that the boundary points A_j and A_{j+1} belong to the line $y = r_k + kx$. Since A_j and A_{j+1} lie above all the other lines $y = r_i + ix$, we have for all i

$$\begin{aligned} r_i + ix_j &\leq r_k + kx_j, \\ r_i + ix_{j+1} &\leq r_k + kx_{j+1}. \end{aligned}$$

From these two estimates it follows that

$$\begin{aligned} |a_i h^i| &\lesssim |a_k h^k| 2^{(k-i)(\log h - x_j)}, \\ |a_i h^i| &\lesssim |a_k h^k| 2^{(i-k)(x_{j+1} - \log h)}. \end{aligned}$$

All in all,

$$|a_i h^i| \lesssim |a_k h^k| 2^{-|k-i|B}.$$

This estimate clearly implies $|P(h)| \gtrsim |a^k h^k| \gtrsim 2^{kB} \geq 1$, provided that B is large enough. \square

Now if we take out the factor of $2^{-l-jM_1}(y_2 - y_1)$, the expression in the RHS of (6.7) has the form of polynomial in $h = y_2 - y_1$ falling under the scope of the lemma with $r_i = l - \gamma_i$. So we have a set E of the form (6.8) such that

$$|[S(x, y_2) - S(x, y_1)]'_x| \gtrsim 2^{-l-jM_1}(y_2 - y_1) \quad \text{if } y_2 - y_1 \in E.$$

We claim that this implies

$$(6.9) \quad |K(y_2, y_1)| \lesssim 2^{l+jM_1} \lambda^{-1} (y_2 - y_1)^{-1} \quad (y_2 - y_1 \in E).$$

Indeed, this will follow from Lemma 4.1 with $k = 1$, if we prove that there exists a constant C independent of y_1 and y_2 such that for fixed y_1 and y_2

- (1) the number of intervals of monotonicity of $[S(x, y_2) - S(x, y_1)]'_x$ considered as a function of x is less than C ,
 (2) the number of intervals comprising the x -set where $\chi_\gamma(x, y_1)\chi_\gamma(x, y_2)$ is non-zero is less than C .

To show (1), note that $\partial_x^N F(x, y) \neq 0$ on W_Q . It follows that

$$\partial_x^{N+1} [S(x, y_2) - S(x, y_1)] = \int_{y_1}^{y_2} \partial_x^N F(x, y) dy \neq 0.$$

Therefore, $[S(x, y_2) - S(x, y_1)]''_{xx}$ vanishes at most $N - 1$ times, which implies (1).

To show (2), it suffices to check that the number of intervals in the set $\{x | (x, y) \in W_Q, a \leq \sigma_j(x, y) \leq b\}$ is bounded by a constant independent of a and b for each $0 \leq j \leq N - 1$. However, this last statement follows from (6.2).

Unfortunately, to prove the claimed norm estimate for T_γ , we will need still another decomposition taking into account the form of the set E . Namely, for $1 \leq k \leq s$ and an integer n we put

$$\chi_{kn}(y) = \psi(2^{\beta_k}y - n),$$

where $\psi(t)$ is the characteristic function of the interval $[0, 1]$, and consider the operators

$$T_{kn}f(x) = \int e^{i\lambda S(x,y)} \chi_\gamma(x,y) \chi_{kn}(y) f(y) dy.$$

We are going to prove by induction in k that for each n

$$\|T_{kn}\| \lesssim \lambda^{-1/2} (\log \lambda) 2^{l/2} l^{1/2} 2^{jM_1/2}.$$

The statement for $k = s$ implies the required estimate (6.6), since $T_\gamma = \sum_n T_{sn}$, and the sum contains no more than $2^{-\beta_s} \leq C$ terms.

For $k = 1$, we use the kernel of the operator $T_{1n}^* T_{1n}$, which has the form

$$\chi_{1n}(y_1) \chi_{1n}(y_2) K(y_2, y_1),$$

where $K(y_2, y_1)$ is the kernel of $T_\gamma^* T_\gamma$. If this expression is not zero, then $|y_2 - y_1| \leq 2^{\beta_s}$. In view of (6.9), and also because $|K| \lesssim 1$, Lemma 3.2 says that

$$\begin{aligned} \|T_{1n}^* T_{1n}\| &\lesssim \int_0^{2^{\beta_s}} \min(1, 2^{l+jM_1} \lambda^{-1} t^{-1}) dt \\ &\lesssim 2^{l+jM_1} \lambda^{-1} \int_0^{\lambda 2^{\beta_s - l - jM_1}} \min(1, t^{-1}) dt \\ &\lesssim 2^{l+jM_1} \lambda^{-1} \log \lambda, \end{aligned}$$

which is even better by a factor of l than what we need.

The induction step is performed by using the decomposition

$$T_{k+1,n} = \sum_{n'} T_{kn'}.$$

We will need the following variant of the Cotlar–Stein lemma, which can be proved by an easy adaptation of the standard proof given in [11], see e.g. Comech [3], Appendix.

Lemma 6.3. *Let T_i be a family of operators on a Hilbert space H such that*

- (1) $T_i T_{i'}^* = 0$ for $i \neq i'$,
 - (2) $\sum_{i'} \|T_i^* T_{i'}\| \leq C$ with a constant C independent of i .
- Then $\|\sum T_i\| \leq C^{1/2}$.*

We have $T_{kn'} T_{kn''}^* = 0$ for $n' \neq n''$. Let us estimate the sum

$$(6.10) \quad \sum_{n''} \|T_{kn'}^* T_{kn''}\|$$

for a fixed n' . Since both $T_{kn'}$ and $T_{kn''}$ appear in the decomposition of $T_{k+1,n}$, we have $|n' - n''| \leq 2^{\beta_{k+1} - \beta_k}$. Further, the kernel of $T_{kn'}^* T_{kn''}$ has the form

$$\chi_{kn'}(y_1) \chi_{kn''}(y_2) K(y_2, y_1).$$

If this expression is different from zero, then

$$(6.11) \quad 2^{\beta_k} |y_2 - y_1| \in [|n' - n''| - 1, |n' - n''| + 1].$$

Assume first that

$$(6.12) \quad 2^{\alpha_{k+1}-\beta_k} + 1 \leq |n' - n''| \leq 2^{\beta_{k+1}-\beta_k} - 1.$$

Then (6.11) implies $|y_2 - y_1| \in E$, and we can use the estimate (6.9). By Lemma 3.2,

$$\begin{aligned} \|T_{kn'}^* T_{kn''}\| &\lesssim \int_{2^{-\beta_k}(|n'-n''|-1)}^{2^{-\beta_k}(|n'-n''|+1)} 2^{l+jM_1} \lambda^{-1} t^{-1} dt \\ &\lesssim 2^{l+jM_1} \lambda^{-1} |n' - n''|^{-1}. \end{aligned}$$

Therefore the part of the sum (6.10) over n'' satisfying (6.12) is bounded by

$$2^{l+jM_1} \lambda^{-1} \sum_{m=2^{\alpha_{k+1}-\beta_k}}^{2^{\beta_{k+1}-\beta_k}} \frac{1}{m} \lesssim 2^{l+jM_1} \lambda^{-1} (\beta_{k+1} - \alpha_{k+1}) \leq 2^{l+jM_1} \lambda^{-1} l,$$

where we used the fact that by Lemma 6.2 (2) $\beta_1 \geq -Bl$.

However, the number of n'' which do not satisfy (6.12) is bounded by a constant in view of Lemma 6.2 (2), so the corresponding part of (6.10) is bounded by $C \sup \|T_{kn''}\|^2 \lesssim l 2^{l+jM_1} (\log \lambda) \lambda^{-1}$ by the induction hypothesis.

By applying Lemma 6.3, we complete the induction step. The theorem is now proved.

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